Mode pairs in $\mathcal{PT}$-symmetric multimode waveguides

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(Received 8 July 2014; published 17 October 2014)

We study mode properties in multimode optical waveguides with parity-time ($\mathcal{PT}$) symmetry. We find that two guiding modes with successive orders $2m-1$ and $2m$ form a mode pair in the sense that the two components of the pair evolve into the same mode when the loss and gain coefficient increases to some critical values, and they experience $\mathcal{PT}$ symmetry breaking simultaneously. For waveguides that in their conservative limit support an odd number of guiding modes, a new mode with a proper order emerges upon the increase of the gain and loss level, so that it pairs with the already existing highest-order mode and then breaks their $\mathcal{PT}$ symmetry simultaneously. Depending on the specific realizations of $\mathcal{PT}$-symmetric potentials, higher-order mode pairs may experience symmetry breaking earlier or later than the lower-order mode pairs do.

DOI: 10.1103/PhysRevA.90.043833 PACS number(s): 42.25.Bs, 42.82.Et

I. INTRODUCTION

One of the postulates of quantum mechanics is that every physical observable corresponds to a Hermitian operator so that the eigenvalues are guaranteed to be all pure real. However, Bender and co-workers revealed that a non-Hermitian Hamiltonian respecting the so-called parity-time ($\mathcal{PT}$) symmetry can still exhibit an entirely real spectrum [1–3]. By definition, a Hamiltonian is said to be $\mathcal{PT}$ symmetric if it shares a common set of eigenfunctions with the $\mathcal{PT}$ operator. The parity operator $\mathcal{P}$, responsible for spatial reflection, is defined through the operations $P \to -P$, $x \to -x$, while the time-reversal operator $\mathcal{T}$ leads to $P \to -P$, $x \to x$ and to complex conjugate $i \to -i$. Given the fact $\mathcal{TH} = P^2/2 + V^*(x)$, a necessary condition for the Hamiltonian to be $\mathcal{PT}$ symmetric is that the potential function $V(x)$ should satisfy the condition $V^*(-x) = V(x)$. However, the latter is only a necessary condition for $\mathcal{PT}$ symmetry, because the transition to a complex spectrum, which is called $\mathcal{PT}$ symmetry breaking, appears upon the increase of the strength of the imaginary part of the potential $V(x)$.

Optical structures are suggested to be a powerful platform for the implementation of $\mathcal{PT}$ physics [4–8]. Spontaneous $\mathcal{PT}$ symmetry breaking has been experimentally observed in passive [9] and active [10] optical waveguide couplers. Since then various $\mathcal{PT}$-symmetric structures were studied, including nonlinear couplers [11–16], periodic [7,17–22] or truncated [23–25] and defective lattices [26,27], pseudopotentials with $\mathcal{PT}$-symmetric [28,29] or inhomogeneous nonlinear terms [30], as well as mixed linear and nonlinear lattices [31–33].

In this paper, with reference to waveguiding structures that support a large number of localized modes, we put forward the first systematic study on the $\mathcal{PT}$-symmetric properties of higher-order modes. We find that modes with successive order $2m$ and $2m-1$ form a *mode pair* (where m is a positive integer) as the two components of the pair evolve into the same mode when the gain and loss level increases to some critical values, and they simultaneously break their symmetry. Interestingly, in the case when the waveguides in their conservative limit accommodate an odd number of modes, the increase of the gain and loss level creates a new mode whose order is larger by one than the already-existing highest-order mode, with which the new mode pairs and they experience $\mathcal{PT}$ symmetry breaking simultaneously. We also find that, depending on the specific realizations of $\mathcal{PT}$-symmetric potentials, the critical value of the gain and loss coefficient beyond which the symmetry of the higher-order mode pair breaks could be larger or smaller than those of the lower-order mode pairs. It should be noted that the simultaneous symmetry breakup of the fundamental and dipole modes has been reported in Refs. [34,35]; however, generic properties of higher-order modes have not been systematically studied, and the latter is the aim of this paper.

II. MODEL

Let us consider the propagation of a laser beam in a multimode waveguide that can be described by a Schrödinger-like equation for the dimensionless field amplitude q,

$$i\frac{\partial q}{\partial z} = -\frac{1}{2}\frac{\partial^2 q}{\partial x^2} - V(x)q. \tag{1}$$

Here x and z are normalized transverse and longitudinal coordinates, respectively. The complex function $V(x)$ describes the waveguide profile, whose real part represents the landscape of the refractive index, while the imaginary part represents the gain and loss modulations. While other types of waveguide profiles have also been checked in our study, for demonstration purposes, we assume in the following $V(x) = p\exp(-\frac{x^2}{d^2})(1 + i\alpha x)$, namely, a Gaussian waveguide with a balanced gain and loss built into the waveguiding region. The parameters p and d characterize the amplitude and width of the waveguide, respectively, and α is the gain and loss coefficient. The eigenmodes of the complex waveguides can be found numerically by looking for the solution of Eq. (1) in the form of $q(x,z) = w(x)\exp(ibz)$, where $w = w_r + iw_i$ are mode wave functions that are generally complex functions (in the no-gain and no-loss limit, the wave function can be chosen to be pure real), and $b = b_r + ib_i$ are the mode propagation constants. With a proper setting of p and d, the waveguide in its conservative limit supports multiple modes, with the

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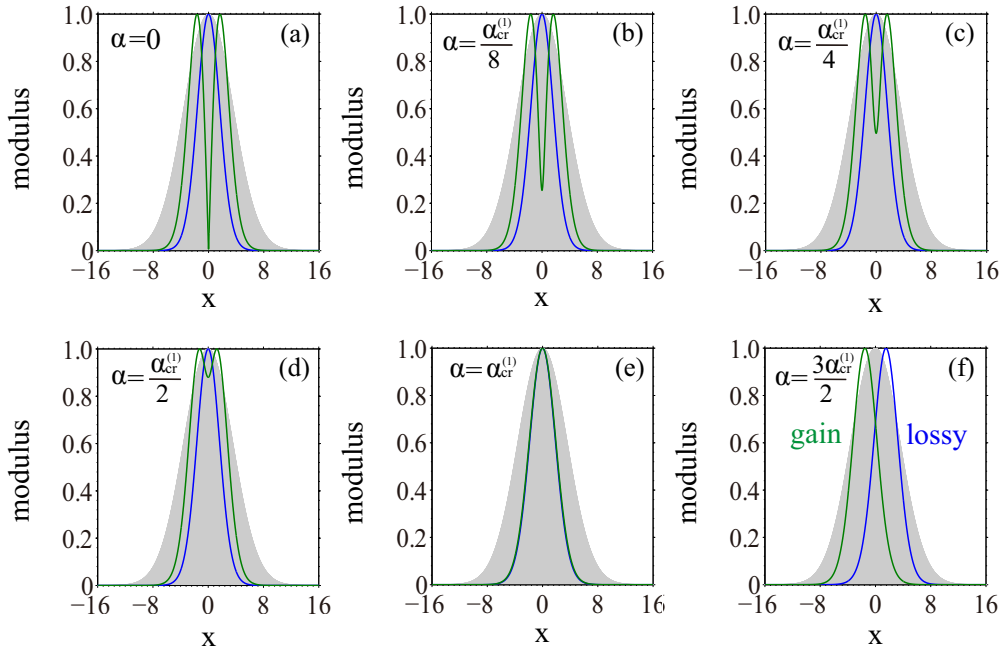

FIG. 1. (Color online) Mode profiles for fundamental and dipole modes at different values of α. The shaded region stands for the landscape of the Gaussian waveguide; $p = 2, d = 5$.

first-order mode being nodeless in profile and the Nth-order mode featuring $N-1$ nodes. Without loss of generality, in the following we set $p = 2$ and $d = 5$, and the corresponding waveguide in its vanishing gain and loss limit supports six modes. We then gradually increase the loss and gain coefficient and watch the evolutions of these modes. The results are presented below.

III. MODE PAIRS

Figure 1 shows the evolutions of fundamental and dipole modes. When $\alpha = 0$, the fundamental mode is bell shaped and the dipole mode features two sharp peaks with a node in between them [Fig. 1(a)]. However, with the increase of α, the valley between the two peaks starts climbing steadily and eventually, at $\alpha = \alpha_{\rm cr}^{(1)}$, the valley vanishes and the dipole mode becomes bell shaped, taking exactly the same shape as the fundamental mode [Fig. 1(e)]! If α increases further, two asymmetric modes occur, with one mostly residing at the lossy region [labeled "lossy" in Fig. 1(f)], and the other mostly at the gain region [labeled "gain" in Fig. 1(f)].

The variations of propagation constants accompanying such mode reshaping are shown in Fig. 2. As expected, the propagation constants of the fundamental and dipole modes remain pure real until the loss or gain level exceeds some critical value. However, with the increase of α, the propagation constants of two modes approach each other, and finally they merge into one at $\alpha = \alpha_{\rm cr}^{(1)}$. This is the point that the dipole evolves into a bell shape and attains the same shape as the fundamental mode. Beyond this point, a pair of complex conjugate b emerges (Fig. 2), corresponding to the gain and lossy mode pair shown in Fig. 1(f).

The approaching of the two components of the mode pairs can also be understood analytically if one performs a perturbation analysis on Eq. (1). The substitution of $q(x,z) = w(x)\exp(ibz)$ into the equation yields the following equation for the stationary function, $w(x)$:

$$-bw + \tfrac{1}{2}w'' + (V_r + i\beta V_r')w = 0, \tag{2}$$

where the prime stands for d/dx, $\beta \equiv -\alpha d^2/2$ is a constant, and $V_r(x)$ is the real part of the potential (its particular form is not important; it is essential that the imaginary part of the potential is proportional to the derivative of the real part). For small α (thus also small β), the solution is looked for perturbatively, as

$$w(x) = w_0(x) + i\beta w_1(x), \tag{3}$$

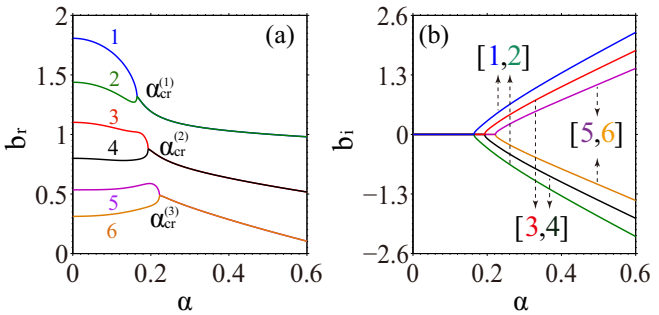

FIG. 2. (Color online) Dependence of the (a) real and (b) imaginary parts of the propagation constants of the first six modes on α; $p = 2, d = 5$.

where functions w_0 and w_1 are real, the zero-order function w_0 satisfies the usual linear Schrödinger equation,

$$-bw_0 + \tfrac{1}{2}w_0'' + V_r(x)w_0 = 0, \tag{4}$$

and the first-order function w_1 satisfies the following inhomogeneous equation:

$$-bw_1 + \tfrac{1}{2}w_1'' + V_r(x)w_1 = -V_r'(x)w_0. \tag{5}$$

Now, applying d/dx to Eq. (4), one obtains

$$-bw_0' + \tfrac{1}{2}(w_0')'' + V_r(x)w_0' = -V_r'(x)w_0. \tag{6}$$

Comparing Eqs. (6) and (5) makes it obvious that

$$w_1(x) \propto w_0'. \tag{7}$$

Thus, while fundamental and dipole modes differ significantly in their shapes (actually they are orthogonal to each other), gradually increasing α from zero introduces an imaginary part into the mode wave functions. Importantly, as Eqs. (3) and (7) show, the imaginary parts of the $\mathcal{PT}$-symmetric mode wave functions are proportional to the first derivative of their real part, with α being the proportional factor. Thus, the fundamental modes that are initially symmetric now gain an imaginary part that is antisymmetric, which is exactly like that of the dipole modes. Similarly, the dipole modes that are initially antisymmetric now gain imaginary parts that are symmetric like fundamental modes. This prediction is well collaborated by numerically results (see Fig. 3). In other words, the increasing imaginary part of the potential leads to the growing weight of the second mode in the field of the mode that was initially "first," and the growing weight of the first mode in the field of the mode that was initially "second." Such modes are not orthogonal any more and actually they start to approach each other. Thus, it is not surprising that,

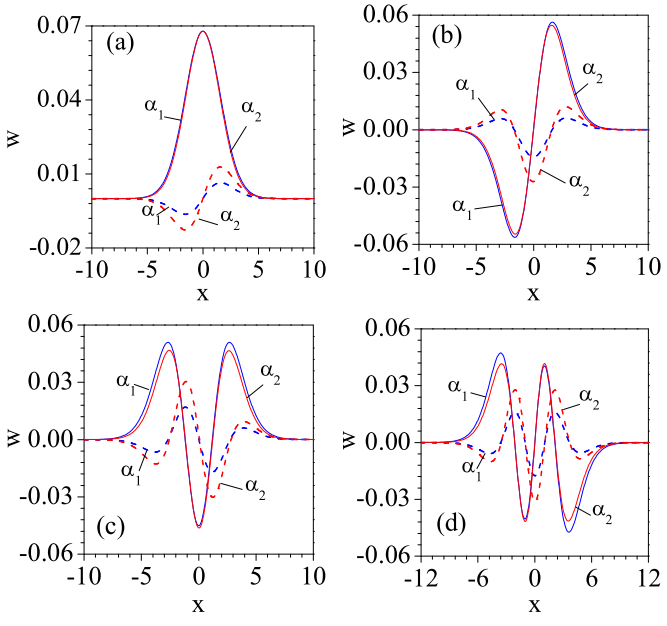

FIG. 3. (Color online) The real (solid lines) and imaginary (dashed lines) parts of the wave functions for the first (a, b) and the second (c, d) mode pairs, at $\alpha_1 = 0.02$ (green lines) and $\alpha_2 = 0.04$ (red lines); $p = 2$, $d = 5$.

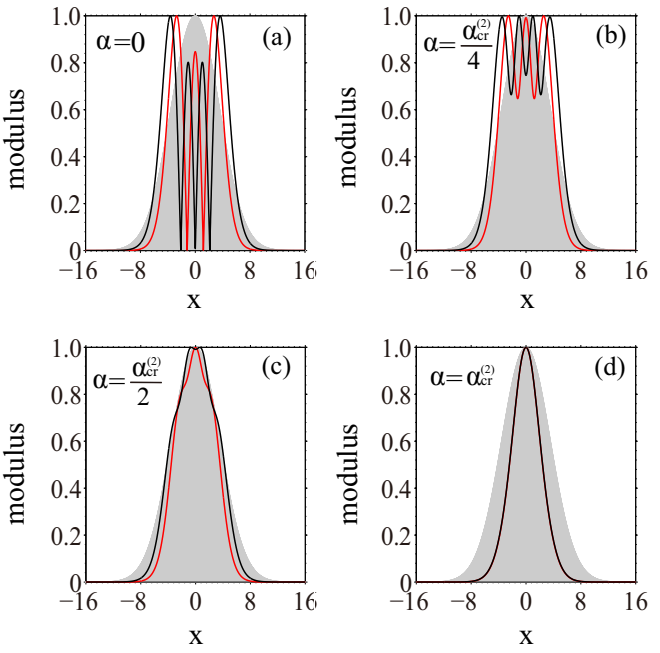

FIG. 4. (Color online) Mode profiles for tripole and quadrupole modes at different values of α; $p = 2$, $d = 5$.

when the imaginary part grows to some critical value, the fundamental and dipole modes take the same profiles and coalesce.

The mode approach and finally simultaneous breakup of $\mathcal{PT}$ symmetry for the dipole and fundamental modes are also observed for tripoles and quadrupoles, for fifth- and sixth-order modes, and so on. These are multihump modes with several valleys between the humps. Interestingly, the increase of α continuously lifts those valleys and weakens the amplitude modulations [Figs. 4(a)–4(c)]. As a result, when α increases to $\alpha_{\rm cr}^{(2)}$, tripoles and quadrupoles evolve into the same bell shape [Fig. 4(d)] and attain the same propagation constant (Fig. 2). The fifth- and sixth-order modes exhibit a similar scenario (Fig. 2). After also performing the analysis on other types of $\mathcal{PT}$-symmetric multimode waveguides, we arrive at a conclusion that, in multimode optical waveguides, guided modes with order $2m$ and $2m-1$ form a *mode pair* in the sense that the two components of the pair evolve into the same mode at $\alpha_{\rm cr}^{(m)}$ and they simultaneously undergo $\mathcal{PT}$ symmetry breaking beyond that point.

IV. MODE PAIRS IN WAVEGUIDES SUPPORTING ODD-NUMBERED MODES

The proposed concept of a mode pair naturally leads to the following question: what happens if a waveguide initially supports an odd number of guided modes, so that its highest-order mode, say, with order $2m-1$, does not have a chance to form a pair? Our thorough studies reveal that the increase of α results in the formation of the $2m$-order mode, which pairs with the already existing $(2m-1)$-order mode, and eventually, like other canonical mode pairs, this new pair enters the symmetry-broken phase. Figure 5 illustrates

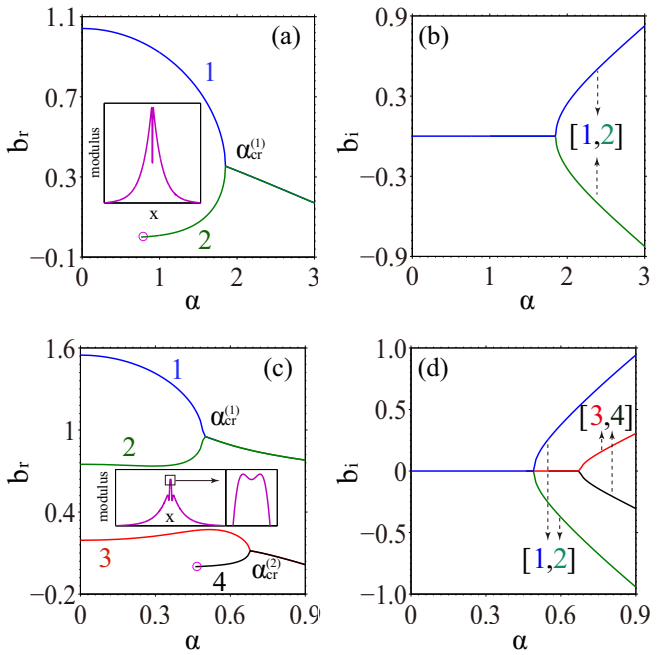

FIG. 5. (Color online) Dependence of mode propagation constants on α for waveguides with (a, b) $p = 2, d = 0.8$ and (c, d) $p = 2$, $d = 2$. Insets in (a) and (c) show the mode profiles corresponding to the circles in the spectrum.

the spectrum when the waveguide in its conservative limit accommodates only one [Fig. 5(a)] and three [Fig. 5(c)] guided modes, respectively. The figure shows that, at the early stage

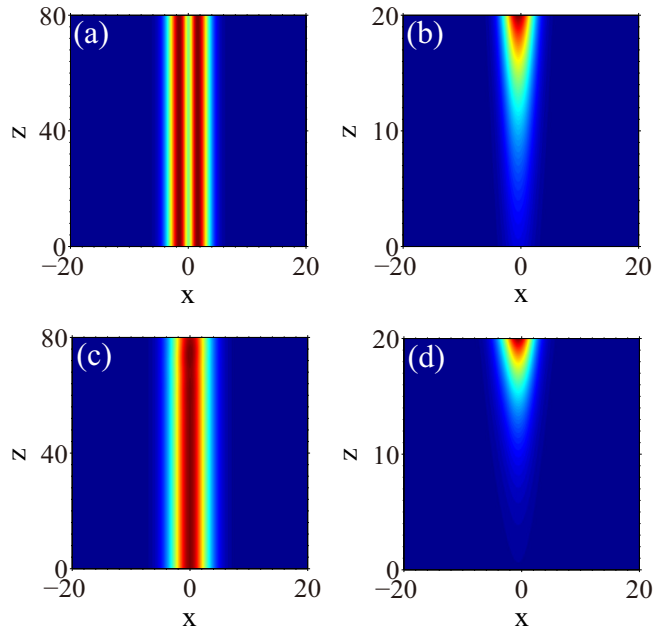

FIG. 6. (Color online) Propagation simulation of the dipole mode at (a) $\alpha = 0.041(<\alpha_{cr}^{(1)})$, and of the mode bifurcating from the fundamental-dipole mode pair at (b) $\alpha = 0.175(>\alpha_{cr}^{(1)})$. Propagation simulation of the quadrupole mode at (c) $\alpha = 0.175(<\alpha_{cr}^{(2)})$, and of the mode bifurcating from the tripole-quadrupole mode pair at (d) $\alpha = 0.22(>\alpha_{cr}^{(2)})$; $p = 2, d = 5$.

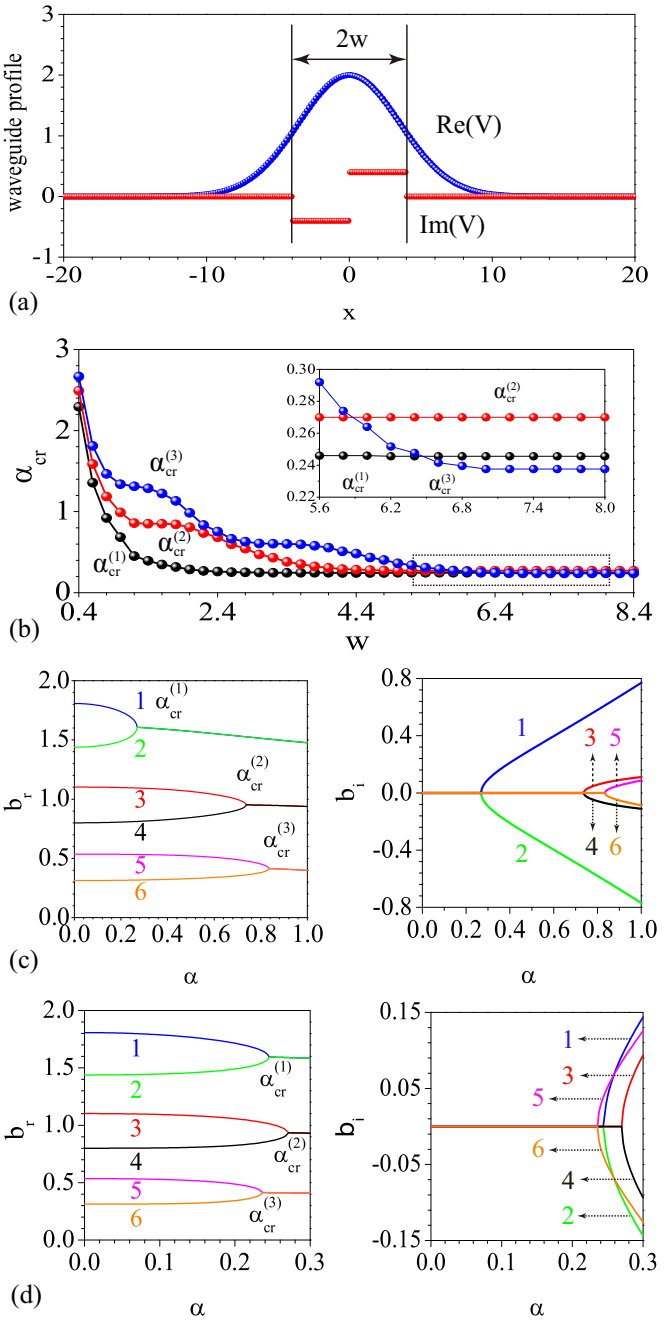

FIG. 7. (Color online) (a) Real [Re(V)] and imaginary [Im(V)] parts of the waveguide profile, with $\mathrm{Re}(V) = p\exp(x^2/d^2)$ for all x, while $\mathrm{Im}(V) = ix/|x|\alpha$ for $x \in (-w, w)$, and 0 otherwise. (b) Dependence of α_{cr} of three mode pairs on w. The inset is the enlargement of the dashed-box portion. The dependence of mode propagation constants on α is shown for (c) $w = 2.2$ and (d) $w = 7.4$. In all the cases, $p = 2, d = 5$.

of the increasing α, the fundamental [Figs. 5(a) and 5(b)] or tripole [Figs. 5(c) and 5(d)] mode evolves by itself without forming a pair with other modes. Interestingly, however, when α increases to some value, a new guided mode featuring two [Fig. 5(a)] or four [Fig. 5(c)] humps appears, which is recognized as a dipole or quadrupole mode. The new mode pairs with the already existing fundamental or tripole mode,

and they experience a simultaneous $\mathcal{PT}$ symmetry breaking with the further increase of α.

V. $\mathcal{PT}$-SYMMETRY-BREAKING POINT OF DIFFERENT MODE PAIRS

Finally we compare the $\mathcal{PT}$-symmetry-breaking point for different mode pairs. In canonical waveguiding geometries such as the Gaussian waveguides considered above, one finds that, with the increase of gain and loss coefficient, the lowest-order mode pair first breaks symmetry, then the higher-order pairs break successively. This property is clearly seen in Figs. 2 and 5, as $\alpha_{\rm cr}^{(1)} < \alpha_{\rm cr}^{(2)} < \alpha_{\rm cr}^{(3)} < \cdots$. Thus, for some specific gain and loss level, while the lower-order mode pairs are already symmetry broken, the higher-order pairs might still remain their symmetry. Figure 6 shows the propagation simulation for fundamental and dipole [Figs. 6(a) and 6(b)] and tripole and quadrupole [Figs. 6(c) and 6(d)] pairs. Note that a same α value ($\alpha = 0.175$) is used in Figs. 6(b) and 6(c). However, as $\alpha_{\rm cr}^{(1)} < 0.175 < \alpha_{\rm cr}^{(2)}$, the modes bifurcating from the fundamental and dipole mode pair experience either amplification [Fig. 6(b)] or decay during propagation, while the modes in the tripole and quadrupole pair propagate in a stationary fashion [Fig. 6(c)]. We note that the fact that the symmetry of the tripole is maintained while that of the two lower-order modes is already broken was indicated in [34].

One might explain that the postponement in the symmetry breaking of the higher-order mode pair is due to the fact that higher-order modes are more spatially extended, and thus the *effective* gain and loss strength they feel is weaker than that of the lower-order pairs; therefore, a larger gain and loss level is required to drive higher-order pairs into symmetry-breaking phases (the effective gain and loss strength is given by the spatially weighted average of the imaginary part of the complex potential over the modal field profile). However, we find that this argument is not always true, and the higher-order mode pairs may also break symmetry *earlier* than lower-order pairs do. A profound example is shown in Fig. 7, where the real part of the potential is still a Gaussian one, while the gain and loss modulation is a step function defined within a finite region with width being $2w$ [Fig. 7(a)]. We examine the dependence of symmetry-breaking points for different mode pairs of the structure on the width of the gain and loss region, w, and the result is shown Fig. 7(b). The figure shows that, when the gain and loss region is narrow (compared with the width of the mode profiles), the breaking points of all mode pairs are nearly the same; with increasing w, it becomes evident that higher-order mode pairs break symmetry later than lower-order mode pairs do [see, for example, Fig. 7(c) for $w = 2.2$]. Interestingly, when w is increased further ($w > 6.4$), the third mode pair is found to first break symmetry, followed by the first mode pair, and then the second mode pair [Fig. 7(d)]. This situation remains true even when $w \to \infty$. Clearly, for such a very spatially extended gain and loss region, the effective gain and loss strengths for all mode pairs are the same, and still, different mode pairs break symmetry at different points. Finally, we should mention that the observed property (the mode pairs of higher orders break symmetry earlier than those of lower orders do) is not caused by the the piecewise nature of the imaginary potential considered in Fig. 6, and a similar picture is observed for other smoothly varying potentials, too.

VI. CONCLUSIONS

We have put forward a systematic study on the properties of $\mathcal{PT}$ symmetry for multimode waveguides. We have revealed that waveguide modes with successive orders $2m-1$ and $2m$ form a mode pair as they gradually evolve into the same mode and experience symmetry breaking simultaneously. For waveguides that support an odd number of guided modes, the increase of the gain and loss coefficient gives birth to a new higher-order mode which pairs with the already existing highest-order mode, and then goes to a symmetry breaking together. Depending on the specific realizations of $\mathcal{PT}$-symmetric potentials, the breaking point of the higher-order mode pair can be later or earlier than those of the lower-order pairs.

ACKNOWLEDGMENTS

The authors thank B. Malomed and Y. Kartashov for useful discussions. F.Y. acknowledges support from the National Natural Scientific Funding of China (Grant No. 11104181).

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